# Analysis of the Changes in Velocity of Balls Rolling Off Edges Using Chi-Squared Model Testing

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ABSTRACT

When a ball rolls over the edge of a table, its horizontal velocity is consistently observed to be higher than the initial horizontal velocity. While multiple studies have investigated this phenomenon, none of these studies provided data on the linear velocity of the ball, nor discussed the variance between the data and the model. This paper studied the motion of a ball rolling over the edge using chi-squared curve fitting and model testing. Video analysis was used to collect velocity data of the ball's motion, and chi-square curve fitting and analysis was performed using a python script. The kick in velocity was observed to decrease as the initial velocity increased, consistent with previous studies. However, the best fit parameter values were not consistent with the derived theoretical model. Furthermore, chi-squared analysis indicated that the best fit model was not a good fit for the data, suggesting that revisions are necessary to the experiment and model.

## Introduction

A typical problem in high school level physics involves a ball rolling off an edge. It is often assumed that the horizontal velocity of the ball remains constant during this motion. However, contrary to assumption, a "kick" is observed, increasing the horizontal component of the ball's velocity.

This phenomenon was discussed by Beeken (2004), who quantitatively described the extra velocity given in the horizontal direction, however, did not provide a phenomenological explanation. This was done by Doucette (2004), who attributed this "kick" to the horizontal component of the normal force exerted on the ball while it is rolling over the edge, and provided a qualitative explanation for the phenomenon. However, in that study, an analysis was not provided on the friction acting on the ball as it rolls off the edge, which was later done by Bacon (2005). In addition, Doucette (2004) did not provide any experimental data of the phenomenon, while both Beeken (2004) and Bacon (2005) included data on the angular velocity of the ball. Despite that, none of these studies provided data on the linear velocity of the ball, nor discussed the variance between the data and the model.

To address the variance between the data and the model, Chi-squared analysis can be applied to data collected from multiple trials. This method has been applied to analyze results from highly precise experiments, including those that have led to the discovery of the Higgs boson (Ivica, 2017), and gravitational waves (Baggio et al., 2000). Unlike ordinary least squares (OLS) and weighted least squares (WLS), chi-squared analysis does not rely on external methods such as correlation coefficients, or F-test; rather it is built on probability theory. Therefore, chi-squared analysis can provide a consistent method to find best fit parameter values to a model, obtain the uncertainties of these values, and gauge the goodness of the best fit provided.

Chi-squared testing is a curve fitting and model testing technique which, given a set of data, can estimate the parameters of a model and evaluate the uncertainties of the model. By including uncertainties, chi-squared testing can not only find the best fit parameter values, but also determine whether the best fit is a good fit for the data. This technique has been applied to analyze results from highly precise experiments, including those that have led to the discovery of the Higgs boson (Ivica, 2017), and gravitational waves (Baggio et al., 2000).

This study aims to generate a dataset of linear velocities of falling balls, develop a model that incorporates uncertainties during the experiment using chi-squared curve fitting and model testing, and evaluate the validity of this approach in falling ball analysis.

Fig 1**.** Experiment setup from the perspective of the slow-motion camera.

## Methods

### Data Collection

#### *Experiment*

A hollow rubber ball of diameter 6 cm was manually rolled over the edge of a flat surface and allowed to fall to the floor (Figure 1). The entire motion was recorded in slow motion by a video camera with a frame rate of 400 fps. The camera was facing perpendicular to the surface of the table. The experiment was repeated five times to produce five independent videos, each of an independent trial of the experiment.

#### *Video Analysis*

The video analysis software Tracker (Brown, 2020) was used to extract position and velocity data from each video. Prior to analysis, each video was prepared and calibrated (Figure 2). The diameter of the ball (6 cm) was used as a calibration stick in each video to provide accurate dimensions for analysis. The frame rate of the video was also set to the frame rate of the camera used (400 fps) to give accurate time measurements. Finally, a coordinate system was included with its origin located at the center of the ball and the positive x-axis pointing horizontally in the direction of the ball's motion.

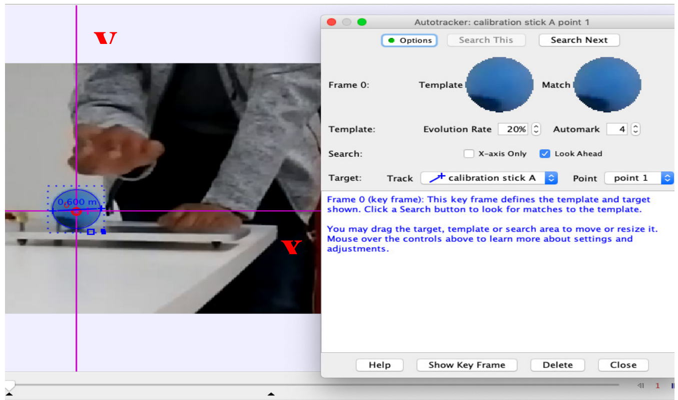


Fig 2. Calibrated video in tracker with calibration stick, coordinate axis and tracking template.

Tracker's auto tracker tool was used to extract position data of the ball over time. The entire ball was selected as a template for the auto tracker, and was tracked until it was out of frame after it fell. Each tracking was then manually inspected to make sure the ball was tracked accurately. If this was not the case, that data would be rejected. Each video was tracked five times for five independent trials.

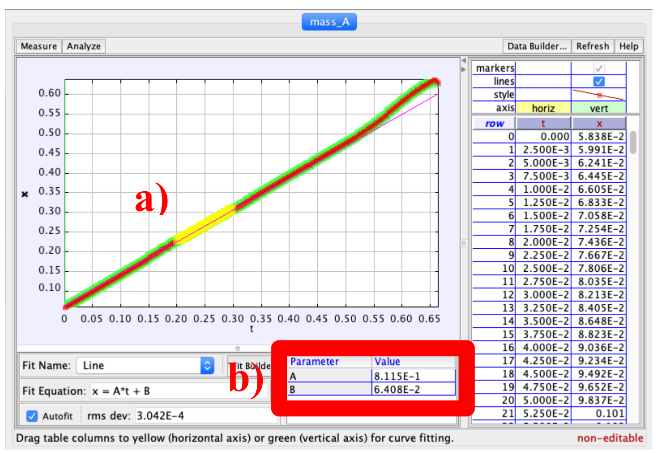


Fig 3**.** Provided x position-time graph of motion from Tracker with a) selected portion of data in yellow; b) best-fit parameter values of selected data. The A value (slope) was taken as the velocity.

After each tracking of the ball, the x (horizontal) component of the ball's velocity was taken from the slope of the x position-time graph (Figure 3). Data points over a portion of both the rolling and falling motion were selected and fitted linearly in Tracker to provide initial and final velocities respectively. Only portions with 15 or more consecutive legible frames were selected. Each video was tracked five times, giving five values of the initial velocity and five values of the final velocity for each video. The initial velocities for each video were then averaged to give one initial velocity and five final velocities per video.

## Data Modelling and Analysis

## Theoretical Model of Velocity

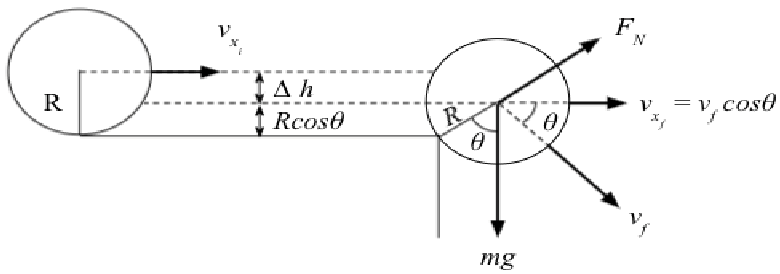


Fig 4. Vectors of the ball rolling over the edge of the table, where R is the radius, $v_f$ is the final velocity, $v_{x_i}$ and $v_{x_f}$ are the respective initial and final horizontal velocities, $\Delta h$ is the change in height of the ball, $\theta$ is the angle in which the ball's center of mass rotates about the edge of the table, $m$ is the mass of the ball, $g$ is the acceleration due to gravity, and $F_N$ is the normal force exerted by the table on the ball.

To analyze the change in velocity of the ball, $v_{x_i}$and $v_{x_f}$ must be found. Because energy is conserved in the system, energy conservation can be used. The initial kinetic energy, $K_1$ of the ball can be represented by equation (1).

$$K_1 = \frac{1}{2}m{v_{x_i}}^2 + \frac{1}{2}I\omega^2 \quad (1)$$

where $I = \beta mR^2$ is the moment of inertia of the ball about its center, $\beta$ is a constant, and $\omega$ is the angular velocity of the ball.

As the ball begins to roll off the edge (Fig. 4), its kinetic energy increases according to equation (2).

$$K_2 = K_1 + mg\Delta h \quad (2)$$

This can also be modeled as an object rotating about the edge of the table by

$$K_3 = \frac{1}{2}I_{edge}{\omega_{edge}}^2 \quad (3)$$

where $I_{edge} = I + mR^2$ by the parallel axis theorem, and $\omega_{edge} = \frac{v_f}{R}$ is the angular velocity about the edge.

By energy conservation, the kinetic energy of the ball rolling off the edge can be equated to the kinetic energy of the ball rotating off of the edge, $K_2 = K_3$. Substituting previous equations for $K_2$ and $K_3$, the above equation becomes

$$\frac{1}{2}m{v_{x_i}}^2 + \frac{1}{2}I\omega^2 + mg(R - Rcos\theta) = \frac{1}{2}(I + mR^2)(\frac{v_f}{R})^2 \quad (4)$$

Substituting $I = \beta mR^2$ and isolating $v_f$, equation (4) can be simplified into the following equation

$${v_f}^2 = \frac{2g}{\beta+1}(R - Rcos\theta) + {v_{x_i}}^2 \quad (5)$$

$v_{x_f}$ when the ball leaves the edge can be expressed as $v_{x_f} = v_f cos\theta$. $cos\theta$ when the ball leaves the edge can be calculated using Newton's 2nd law.

$$mgcos\theta - F_N = \frac{m{v_f}^2}{R} \quad (6)$$

When the ball leaves the edge, $F_N = 0$. Thus, $cos\theta$can be expressed as

$$cos\theta = \frac{{v_f}^2}{gR} \quad (7)$$

The final x velocity of the ball can then be given by

$$v_{x_f} = \frac{{v_f}^3}{gR} \quad (8)$$

Substituting the values found in equations (7) and (8) into equation (5), the x component of the final velocity can be linearly related to the x component of the initial velocity in equation (9) in the form $y = Ax + B$

$$v_{x_f}{}^{2/3} = \frac{(\beta+1)}{(\beta+3)(gR)^{2/3}} v_{x_i}{}^{2} + \frac{2(gR)^{1/3}}{(\beta+3)} \quad (9)$$

where $v_{x_f}{}^{2/3}$ is the dependent variable and $v_{x_i}{}^{2}$ is the independent variable. Substituting $\beta = \frac{2}{3}$ for a hollow sphere with negligible thickness, $R = 0.03m$ and $g = 9.81\frac{m}{s^2}$ into equation (9), the parameters can be found for the ball used.

$$v_{x_f}{}^{2/3} = 1.03 v_{x_i}{}^{2} + 0.36 \quad (10)$$

Along with chi-squared analysis, best fit parameters can be compared to the predicted parameters above to determine the accuracy of the model.

*2.2.2. Chi-squared testing*

Chi-squared testing is a curve fitting and model testing technique which evaluates the probability of obtaining a set of data points, which can be expressed as

$$P(data) \propto e^{-\frac{\chi^2}{2}} \quad (11)$$

such that

$$\chi^2 = \sum_{i=1}^{N} \frac{(y_i - y_{model_i})^2}{\sigma^2} \quad (12)$$

where *N* is the number of data points, *y* is the dependent variable and $\sigma$ is the standard error of *y* (Witkov & Zengel, 2019). The greatest probability of generating the data occurs at chi-squared min. By minimizing $\chi^2$, the best fit parameters can be found.

However, in order for this fit to be a good fit, data points should be within one standard error of the best fit line. Thus, substituting $y_i = y_{model} \pm \sigma_i$ into equation (12) gives a good fit range of chi-squared min less than or within the order of N.

A python script (Witkov, 2019) was used to perform chi-squared curve fitting and analysis on the data collected. The data was fit to a 2-parameter linear model in the form of y = Ax + B. The average initial velocity for each video was inputted into the array ‘ind_var’ while the final velocities for each of the five videos were inputted into their respective numbered ‘dep_var’ arrays. The average of each ‘dep_var’ array would then be calculated, giving five initial – final velocity pairs, or $N = 5$ data points. Chi-squared min would then be found to obtain the best fit values of A and B. The script would output the best fit values of A and B, the value of chi-squared min, the number of data points (*N*), and the good fit range of chi-squared, which was given by $N \pm \sqrt{2N}$.

## Results

In all trials, the horizontal velocity of the ball was observed to increase after the ball rolls over the edge. In addition, the amount of velocity increase appears to be dependent on the initial velocity. At relatively low velocities, the horizontal velocity of the ball was observed to increase significantly after the ball rolled over the edge (Table 1).

Table 1.Data collected for videos analyzed through Tracker(velocity is in *m/s*).Standard error of the final velocity given as $\sigma$

| Video | $v_{x_i}$ | $v_{x_f}$ | $\overline{v_{x_i}}$ | $\overline{v_{x_i}}$ | $\Delta\overline{v_x}$ | $\sigma$ |
|---|---|---|---|---|---|---|
| #1 | 0.21 | 0.41 | 0.26 | 0.45 | 0.20 | 0.020 |
| | 0.27 | 0.50 | | | | |
| | 0.27 | 0.43 | | | | |
| | 0.27 | 0.50 | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0.26 | 0.42 | | | | |
| | 0.47 | 0.57 | | | | |
| | 0.44 | 0.50 | | | | |
| #2 | 0.44 | 0.51 | 0.44 | 0.52 | 0.08 | 0.018 |
| | 0.42 | 0.55 | | | | |
| | 0.42 | 0.47 | | | | |
| | 0.29 | 0.41 | | | | |
| | 0.19 | 0.39 | | | | |
| #3 | 0.21 | 0.40 | 0.22 | 0.40 | 0.17 | 0.004 |
| | 0.22 | 0.39 | | | | |
| | 0.21 | 0.40 | | | | |
| | 0.99 | 1.01 | | | | |
| | 0.99 | 1.01 | | | | |
| #4 | 0.97 | 1.00 | 0.99 | 1.01 | 0.03 | 0.009 |
| | 0.99 | 1.05 | | | | |
| | 1.00 | 1.00 | | | | |
| | 0.26 | 0.38 | | | | |
| | 0.26 | 0.37 | | | | |
| #5 | 0.26 | 0.38 | 0.26 | 0.37 | 0.11 | 0.012 |
| | 0.26 | 0.38 | | | | |
| | 0.26 | 0.32 | | | | |

Performing chi-squared analysis on this data set produced a chi-squared min value of 22.60, with $N = 5$ and a chi-squared good fit range of [1.84, 8.16]. The best fit parameter values were found to be $A = 0.50$ and $B = 0.52$. Figure 5 plots $v_{x_i}$ vs $v_{x_f}$ with the best fit model and the theoretical model derived in equation (10), where $A = 1.03$ and $B = 0.36$. Errors for $v_{x_f}^{\frac{2}{3}}$ were given by $\sigma$ (Table 1).

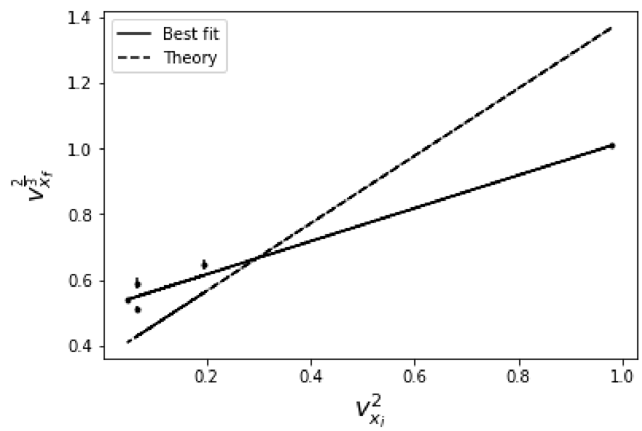


Fig 5. Comparison between chi-squared best fit parameters and theoretical model derived in equation (10) of $v_{x_f}{}^{2/3}$ vs $v_{x_i}{}^{2}$.

Figure 6 shows a 68% and 95% confidence interval contour graph of A and B, corresponding to parameter values within $\chi^2{}_{min} + 2.3$and $\chi^2{}_{min} + 6$ respectively.

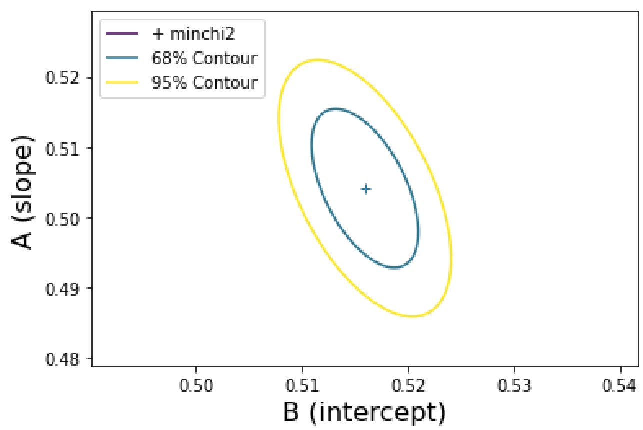


Fig 6. Two parameter graph of 68% and 95% confidence level contour lines centered on the best fit parameters of A and B (0.50, 0.52).

## Discussion and Conclusion

This study represents the first research on applying chi-squared analysis on the classic ball over the edge problem in physics. By applying this novel method, uncertainties during the experiments were captured in this model. The velocity of the ball was observed to increase for all initial velocities after rolling over the edge, consistent with results from previous studies (Doucette, 2004; Bacon, 2005). Results suggested that the amount of increase in velocity is a function of initial velocity. It was observed that for initial velocities below 0.30 m/s, the velocity would increase by 0.10-0.20 m/s, while for an initial velocity of 0.99 m/s, the velocity increased by an average of only 0.03 m/s. This observation is consistent with the conclusions made by Bacon (2005) that the “kick” seems to disappear at higher velocities. A likely reason for this is a difference in the motion experienced by the ball between lower and higher velocities. As derived in 2.2.1, the “kick” in velocity is due to the coupling of rotation and translation from rolling friction. At higher initial velocities, rolling shifts to sliding, thus the derivation of the model is no longer accurate.

Best fit parameter results showed significant differences between the predicted model and the data obtained. The true parameter values of $A = 1.03$ and $B = 0.36$ were well beyond the 95% contour of the best fit values of $A = 0.50$ and $B = 0.52$ (Figure 6), thus the model should be rejected. These discrepancies may have been due to incorrect assumptions made in the derivation of the model. For instance, rolling resistance is neglected in the derived model, which could have resulted in an inaccurate prediction of the final velocity of the ball. This is supported by the fact that the increased velocity of the ball over the edge only occurs if the ball is moving at a slow speed, which is when rolling resistance is greatest. However, testing of a larger variety of initial velocities could provide a better idea of the relationship between the initial and final velocities, and how the "kick" changes as a function of initial velocity.

Results from chi-squared fit indicated that the best fit was not a good fit for this data. The minimum chi-squared value for the data was 22.60, which is outside the chi-squared good fit range of [1.84, 8.16] for $N = 5$. It was also noticed that the standard errors were relatively small in all cases, which would have caused the minimum chi-squared value to increase. This may have resulted from limitations of the method used, producing underestimated uncertainties and thus smaller standard errors. The usage of video analysis and the videos themselves are limited to one field of view, which could underestimate the uncertainties. Given the nature of video tracking, different tracking of the same video would likely result in very similar readings, thus decreasing the standard error. Furthermore, only one trial was done for each initial velocity. Uncertainties would be better represented by conducting multiple trials for the same initial velocity. For this, the ball would need to be mechanically given a set initial velocity, which can be done using a ramp as done by Bacon (2005).

The usage of chi-squared analysis on this problem has uncovered limitations on the experiment and model that would not have been revealed by other curve fitting and data analysis methods. By using probability and including uncertainties in the analysis, a more accurate fit to the model can be generated, the goodness of fit of the model could also be tested, and the value of the minimum chi-squared provides a gauge for model improvement. Other methods such as linear regression and ordinary least squares do not include uncertainties, nor a method to gauge whether the best fit is a good fit. Due to the sensitive nature of chi-squared model testing, model rejection was not unexpected, and simply indicated necessary revisions to the experiment or model.

This study has proposed that the differences between the analytical solution and the experiment were caused by a shift from rolling to slipping at higher velocities. Future studies could focus on testing this hypothesis by performing additional experiments with technologies like digital imaging correlation (DIC). In this way, the amount of rotation and sliding can be quantified and an analytical solution accounting for those can be derived. Applying chi-squared analysis to this new dataset could reveal the underlying mechanism of the kick in velocity.

To the best of the author's knowledge, this is the first study where chi-square analysis has been applied to the problem of the ball rolling over the edge. This study provides a new dataset for the initial and final linear velocities of a ball rolling off the edge, which has not been investigated in prior studies. The results of this study have reaffirmed observations made in prior studies, and have revealed limitations to the model and experiment.

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